2. Numbers and Sequences

Introduction for Exercise 2.1

Concept corner

Theorem 1 - Euclid's division Lemma : Let a and b $(a > b)$ be any two positive integers. Then, there exist unique integers q and r such that $a = bq + r, 0 \leq r < b$.

Note:

- The remainder is always less than the divisor.
- If $r = 0$ then $a = bq$ so b divides a.
- Conversely, if b divides a then $a = bq$

Generalised form of Euclid's division lemma:

If a and b are any two integers then there exist unique integers q and r such that $a = bq + r$, where $0 \leq r < |b|$

Theorem 2: If a and b are positive integers such that $a = bq + r$, then every common divisor of a and b is a common divisor of b and r and vice – versa.

Euclid's Division Algorithm :

To find Highest Common Factor of two positive integers a and b where $a > b$.

Step-1: Using Euclid's division lemma $a = bq + r; 0 \leq r < b$ where q is the quotient , r is the remainder if $r = 0$ then b is the Highest Common Factor of a and b.

Step-2: Otherwise applying Euclid's division lemma divide b by r to get $b = rq_1 + r_1,$ $0 \leq r_1 < r$.

Step-3: If $r_1 = 0$ then r is the highest common factor of a and b.

Step-4: Otherwise using Euclid's division lemma repeat the process until we get the remainder zero. In that case, the corresponding divisor is the HCF of a and b.

Note:

- The above algorithm will always produce remainder zero at some stage. Hence the algorithm should terminate.
- Euclids division algorithm is a repeated application of division lemma until we get zero remainder.
- Highest Common Factor (HCF) of two positive numbers is denoted by (a, b)
- Highest Common Factor (HCF) is called as Greatest Common Divisor (GCD)

Theorem 3: If a, b are two positive integers with $a > b$ then GCD of (a, b) =GCD of $(a - b, b)$.

Highest common factor of three numbers : Let a, b, c be the given positive integers

(i) Find HCF of a, b call it as d, $d = (a, b)$

(ii) Find HCF of d and c.

This will be the HCF of the three given numbers a, b, c.

Note: Two positive integers are said to be relatively prime or co prime if their highest common factor is 1

Introduction for Exercise 2.2

Concept corner

Fundamental Theorem of Arithmetic:

Every natural number except 1 can be factorized as a product of primes and this factorization is unique except for the order in which the prime factors are written.

In general, we conclude that given a composite number N, we decompose it uniquely in the form $N = p_1^{q_1} \times p_2^{q_2} \times p_3^{q_3} \times \ldots \times p_n^{q_n}$ Where $p_1, p_2, p_3, \ldots p_n$ are primes and $q_1, q_2, q_3 \ldots q_n$ are natural numbers.

Significance of the Fundamental Theorem of Arithmetic:

- If a prime number p divides ab then either p divides a or p divides b. That is p divides at least one of them.
- If a composite number n divides ab, then n neither divide a nor b.

  For example, 6 divides 4×3 but 6 neither divide 4 nor 3.

Introduction for Exercise 2.3

Concept corner

Modular Arithmetic: Modular Arithmetic is a system of arithmetic for integers where numbers wrap around a certain value.

Congruence Modulo: Two integers a and b are congruence modulo n if they differ by an integer multiple of n. That $b - a = kn$ for some integer k. This can also be written as $a \equiv b \pmod{n}$

Here, the number n is called modulus. In other words,

$a \equiv b\ (mod\ n)$ means $a - b$ is divisible by n. [dividend = remainder (mod divisor)]

Example: $61 \equiv 5\ (mod\ 7)$ because $61 - 5 = 56$ is divisible by 7.

Note:

- When a positive integer is divided by n, then the possible remainder are $0,1,2,3, \ldots n-1$
- Thus, when we work with modulo n, we replace all the numbers by their remainders upon division by n, given by $0,1,2,3, \ldots, n-1$.

Connecting Euclid's Division lemma and Modular Arithmetic.

Let m and n be integers, where m is positive. Then by Euclid's division lemma, we can write $n = mq + r$ were $0 \leq r < m$ and q is an integer,

$$n = mq + r$$
$$n - r = mq$$
$$n - r \equiv 0 (mod\ m)$$
$$n \equiv r\ (mod\ m)$$

Thus the equation $n = mq + r$ through Euclid's Division lemma can also be written as $n \equiv r\ (mod\ m)$.

Note: Two integers a and b are congruent modulo m, written as $a \equiv b\ (mod\ m)$, if they leave the same remainder when divided by m.

Modulo operations:

Theorem 5 : a, b, c and d are integers and m is a positive integer such that if $a \equiv b\ (mod\ m)$ and $c \equiv d\ (mod\ m)$ then (i) $(a + c) \equiv (b + d)(mod\ m)$

(ii) $(a - c) \equiv (b - d)(mod\ m)$ (iii) $(a \times c) \equiv (b \times d)(mod\ m)$

Theorem 6 : If $a \equiv b\ (mod\ m)$then (i) $ac \equiv bc\ (mod\ m)$

(ii) $a \pm c \equiv b \pm c\ (mod\ m)$for any integer c.

Note: While solving congruent equations, we get infinitely many solutions compared to finite number of solutions in solving a polynomial equation in Algebra.

Introduction for Exercise 2.4

Concept corner

Sequences : A real valued sequence is a function defined on the set of natural numbers and taking real values.

Term : Each element in the sequence is called a term of the sequence.

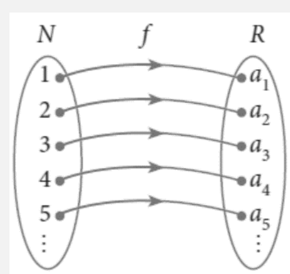
Finite sequence:

If the number of elements in a sequence is finite then it is called a Finite sequence.

Infinite sequence:

If the number of elements in a sequence is infinite then it is called an Infinite sequence.

Sequence as a function:



A sequence can be considered as a function defined on the set of natural numbers N. In particular a sequence is a function $00f: N \rightarrow R$, where R is the set of all real numbers.

If the sequence is of the form a_1, a_2, a_3 , ... then we can associate the function to the sequence to the sequence $a_1, a_2, a_3 \ldots$ by $f(k) = a_k$, $k = 1,2,3, \ldots$

Note: Though all the sequences are functions not all the functions are sequences.

Introduction for Exercise 2.5

Concept corner

Arithmetic progression: Let a and d be real numbers. Then the numbers of the form $a, a+d$, $a+2d, a+3d, a+4d, \ldots$ is said to form Arithmetic progression denoted by A.P. The number $'a'$ is called the first term and $'d'$ is called the common difference.

(i) The General form	$a, a+d, a+2d, a+3d, \ldots \quad a+(n-1)d$.
(ii) n^{th} term	$t_n = a+(n-1)d$
(iii) Common difference	$d = t_2 - t_1 = t_3 - t_2 = t_4 - t_3 = \cdots$ $d = t_n - t_{n-1}$ for $n = 2,3,4,\ldots$ The common difference of an $A.P$ can be positive, negative or zero
(iv) Total number of terms	$n = \left(\frac{l-a}{d}\right) + 1$

Note:

- The difference between any two consecutive terms of an $A.P.$ is always constant. That constant value is called the common difference.
- If there are finite numbers of terms in an $A.P$ then it is called Finite Arithmetic progression. If there are infinitely many terms in an A.P. then it is called Infinite Arithmetic progression.
- An Arithmetic progression having a common difference of zero is called a **constant arithmetic progression**
- In a finite $A.P$ whose first term is a and last term l, then the number of terms in the $A.P$ is given by $l = a+(n-1)d$ gives $n = \left(\frac{l-a}{d}\right) + 1$

In an Arithmetic progression

- If every term is added or subtracted by a constant, then the resulting sequence is also an $A.P$.
- If every term is multiplied or divided by a non- zero number, then the resulting sequences is also an A.P.
- If the sum of three consecutive terms of an A.P is given, then they can be taken as $a-d, a$ and $a+d$. Here the common difference is d.
- If the sum of four consecutive terms of an $A.P$ is given then, they can be taken as $a-3d, a-d, a+d$ and $a+3d$. Here common difference is $2d$.

Condition for three numbers to be in $A.P$.

Three non zero numbers a, b, c are in $A.P$ if and only if $2b = a + c$

Introduction for Exercise 2.6

Concept corner

Series	The sum of the terms of a sequence is called series. Let $a_1, a_2, a_3, \dots a_n \dots$ be the sequence of real numbers. Then the real number $a_1 + a_2 + a_3 + \cdots$ is defined as the series of real numbers.
Finite series	If a series has finite number of terms then it is called a Finite series.
Infinite series	If a series has infinite number of terms then it is called an infinite series.
Arithmetic series	A series whose terms are in Arithmetic progression is called Arithmetic series.

Sum of n terms of an $A.P.$

- The sum of first n terms of a Arithmetic progression denoted by S_n is given by,

$$S_n = a + (a+d) + (a+2d) + \cdots + a + (n-1)d = \frac{n}{2}[2a + (n-1)d]$$

- If the first term a, and the last term $l(n^{th}\ term)$ are given then, $S_n = \frac{n}{2}(a + l)$

Introduction for Exercise 2.7

Concept corner

Geometric progression

Definition	A Geometric progression is a sequence in which each term is obtained by multiplying a fixed non-zero number to the preceding term except the first term. The fixed number is called common ratio. The common ratio is usually denoted by r.
General form	Let a and $r \neq 0$ be real numbers. $a, ar, ar^2, \dots ar^{n-1}$ is called General form of $G.P.$ a is called first term, r is called common ratio.
General term	$t_n = a\,r^{n-1}$

☑ If we consider the ratio of successive terms of the $G.P.$ then we have.

$$\frac{t_2}{t_1} = \frac{ar}{a} = r\ ;\ \frac{t_3}{t_2} = \frac{ar^2}{ar} = r\ ;\ \frac{t_4}{t_3} = \frac{ar^3}{ar^2} = r\ ;\ \frac{t_5}{t_4} = \frac{ar^4}{ar^3} = r$$

Thus the ratio between any two consecutive terms of the Geometric progression is always constant and that constant is the common ratio of the given progression.

☑ When the product of three consecutive terms of a G.P. are given, we can take the three terms as $\frac{a}{r}, a, ar$.

☑ When the products of four consecutive terms are given for a $G.P.$ then we can take the four terms as $\frac{a}{r^3}, \frac{a}{r}, ar, ar^3$

☑ When each term of a Geometric progression is multiplied or divided by a non – zero constant then the resulting sequence is also a Geometric progression.

Condition for three numbers to be in G. P.

Three non – zero numbers a, b, c are in $G.P.$ if and only if $b^2 = ac$

Total amount for compound interest is

$$A = P\left(1 + \frac{r}{100}\right)^n$$

Where, A is the amount, P is the principal, r is the rate of interest and n is the number of years.

Introduction for Exercise 2.8

Concept corner

Geometric Series: A series whose terms are in Geometric progression is called Geometric series.

Sum to n terms of a G.P:

Sum to n terms of a G.P is $S_n = a + ar + ar^2 + ar^3 + \cdots + ar^{n-1}$

$r \neq 1, r > 1$	$S_n = a\left(\frac{r^n-1}{r-1}\right)$
$r = 1$	$S_n = a + a + a + \cdots + a$ $S_n = na$
$r < 1$	$S_n = a\left(\frac{1-r^n}{1-r}\right)$

The sum of infinite terms of a G.P is given by $a + ar + ar^2 + \cdots = \frac{a}{1-r}$, $-1 < r < 1$

Introduction for Exercise 2.9

Concept corner

Special Series: There are some series whose sum can be expressed by explicit formulae. Such series are called special series.

Sum of first n natural numbers	$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$
Sum of first n odd natural numbers	$1 + 3 + 5 + \cdots + (2n-1) = \frac{n}{2} \times 2n = n^2$
Sum of squares of first n natural numbers	$1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$
Sum of cubes of first n natural numbers	$1^3 + 2^3 + 3^3 + \cdots + n^3 = \left[\frac{n(n+1)}{2}\right]^2$

- Sum of divisors of one number excluding itself is the other. Such pair of numbers is called **Amicable numbers** or **Friendly Numbers**
- The sum of first n natural numbers are also called '**Triangular Numbers**' because they form triangle shapes.
- The sum of squares of first n natural numbers are also called **Square Pyramidal Numbers** because they form pyramid shapes with square base.